ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 218 (2008) 259–269
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Bilevel multiplicative problems: A penalty approach to optimality and a cutting plane based algorithm☆

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Received 29 September 2006; received in revised form 29 December 2006

Abstract

Bilevel programming has been proposed for dealing with decision processes involving two decision makers with a hierarchical structure. They are characterised by the existence of two optimisation problems in which the constraint region of the upper level problem is implicitly determined by the lower level optimisation problem. In this paper we focus on the class of bilevel problems in which the upper level objective function is linear multiplicative, the lower level one is linear and the common constraint region is a bounded polyhedron. After replacing the lower level problem by its Karush–Kuhn–Tucker conditions, the existence of an extreme point which solves the problem is proved by using a penalty function approach. Besides, an algorithm based on the successive introduction of valid cutting planes is developed obtaining a global optimal solution. Finally, we generalise the problem by including upper level constraints which involve both level variables.

MSC: 90C26; 90C30

Keywords: Bilevel programming; Multiplicative programming; Penalty approach; Cutting plane

1. Introduction

Bilevel problems consist in determining a vector $x = (x_1, x_2) \in \mathbb{R}^n$ such that

$$\begin{aligned} \min_{(x_1,x_2)\in S} \quad & f_1(x_1, x_2) \\ \text{s.t.} \quad & x_2 \in \operatorname*{argmin}_{v\in S(x_1)} f_2(x_1, v), \end{aligned} \tag{1}$$

where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$ are the variables controlled by the upper level and the lower level decision maker, respectively; $f_1, f_2 : \mathbb{R}^n \longrightarrow \mathbb{R}$, $n = n_1 + n_2$; $S \subset \mathbb{R}^n$ defines the common constraint region and $S(x_1) = \{x_2 \in \mathbb{R}^{n_2} : (x_1, x_2) \in S\}$.

These mathematical programs provide an appropriate model to hierarchical decision processes with two decision makers, the leader and the follower, each controlling part of the variables and having his/her own objective function.

☆ This research work has been supported by the Spanish Ministry of Education and Science under Grant MTM2004-00177.
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0377-0427/$ - see front matter
doi:10.1016/j.cam.2007.01.011

Bilevel problems have received increasing attention in literature. Bard [2] and Dempe [8] are good general references on this topic. Due to its structure, they are nonconvex and quite difficult to deal with, even when all functions involved are linear. One of the main characteristics of bilevel problems is that, unlike general mathematical problems, the bilevel problem may not possess a solution even when f_1 and f_2 are continuous and S is compact. In order to make sure that the bilevel problem is well posed, it is usually assumed that, for each value of the upper level variables x_1, there is a unique solution of the lower level problem.

In this paper we focus on the class of bilevel problems in which the upper level objective function f_1 is linear multiplicative, the lower level one f_2 is linear and the common constraint region S is a bounded polyhedron (LMLB problem). By using a penalty function approach, we prove that there is an extreme point of S which solves the problem. Moreover, an algorithm based on the successive introduction of valid cuts is developed obtaining a global optimal solution. To the best knowledge of the authors this is the first time that multiplicative bilevel problems are considered. Nevertheless, when only one level of decision exists, multiplicative programming has been extensively studied because of the large number of practical applications including microeconomics, financial optimisation, plant layout design and multicriteria optimisation problems [4,11]. The paper is organised as follows. Section 2 provides the main theoretical results on optimality. In Section 3, taking the idea of valid cuts from Horst and Tuy [9], an algorithm is developed which provides a global optimal solution and there is an example which illustrates its application. Section 3 analyses LMLB problems which include upper level constraints involving both level variables. Finally, Section 5 concludes the paper with final remarks and future work.

2. Optimality properties

Using the common notation in bilevel programming, the LMLB problem can be stated as

$$\begin{aligned} \min_{x_1} \quad & (\alpha + c_{11}x_1 + c_{12}x_2)(\beta + c_{21}x_1 + c_{22}x_2) \quad \text{where } x_2 \text{ solves} \\ \min_{x_2} \quad & d_2x_2 \\ \text{s.t.} \quad & A_1x_1 + A_2x_2 \leqslant b, \\ & x_1 \geqslant 0, \quad x_2 \geqslant 0, \end{aligned} \tag{2}$$

where $\leqslant, \geqslant$ is understood componentwise; $A_1 : m \times n_1$; $A_2 : m \times n_2$; $d_2, c_{11}, c_{12}, c_{21}, c_{22}, b$ are vectors of conformal dimension; α, β are scalars and $\alpha + c_{11}x_1 + c_{12}x_2 > 0$, $\beta + c_{21}x_1 + c_{22}x_2 > 0$, $\forall (x_1, x_2) \in S$, the polyhedron defined by the constraints. We assume that S is a nonempty and bounded polyhedron of full dimension in $\mathbb{R}^n$.

Let S_1 be the projection of S onto $\mathbb{R}^{n_1}$. For each $x_1 \in S_1$, a feasible solution to the LMLB problem is obtained by solving the following linear programming problem:

$$\begin{aligned} \mathrm{LP}(x_1): \quad \min_{x_2} \quad & d_2x_2 \\ \text{s.t.} \quad & A_2x_2 \leqslant b - A_1x_1, \\ & x_2 \geqslant 0. \end{aligned} \tag{3}$$

Let $M(x_1)$ be the set of optimal solutions to (3), we assume that it is nonempty and a singleton. Refs. [2,8] show the difficulties which may arise when $M(x_1)$ is not single-valued. The feasible region of problem (2), called inducible region, is implicitly defined as follows:

$$\mathrm{IR} = \{(\tilde{x}_1, \tilde{x}_2) : \tilde{x}_1 \geqslant 0, \ \tilde{x}_2 = \operatorname{argmin}\{d_2x_2 : A_2x_2 \leqslant b - A_1\tilde{x}_1, x_2 \geqslant 0\}\}.$$

Let us consider the linear multiplicative problem:

$$\begin{aligned} \min_{x_1, x_2} \quad & (\alpha + c_{11}x_1 + c_{12}x_2)(\beta + c_{21}x_1 + c_{22}x_2) \\ \text{s.t.} \quad & A_1x_1 + A_2x_2 \leqslant b, \\ & x_1 \geqslant 0, \quad x_2 \geqslant 0. \end{aligned} \tag{4}$$

If an optimal solution $(\tilde{x}_1, \tilde{x}_2)$ of (4) pertains to IR, then it is an optimal solution of the LMLB problem. In general, this will not be true, since both decision makers usually have conflicting objectives.

Theorem 1. *A point (x_1^*, x_2^*) is an optimal solution of the problem* (2) *iff there exists $u^* \in \mathbb{R}^m$ such that (x_1^*, x_2^*, u^*) is an optimal solution of the following one level problem*:

$$\begin{aligned} \min_{x_1,x_2,u} \quad & (\alpha + c_{11}x_1 + c_{12}x_2)(\beta + c_{21}x_1 + c_{22}x_2) \\ \text{s.t.} \quad & A_1x_1 + A_2x_2 \leqslant b, \\ & -u^t A_2 \leqslant d_2, \\ & d_2x_2 - u^t(A_1x_1 - b) = 0, \\ & x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad u \geqslant 0. \end{aligned} \tag{5}$$

Proof. Since (2) and (5) have the same objective function, we only need to prove that the feasible regions of both problems are equal. The dual to the linear program (3) is given by

$$\begin{aligned} \mathrm{DLP}(x_1): \quad \max_{u} \quad & u^t(A_1x_1 - b) \\ \text{s.t.} \quad & -u^t A_2 \leqslant d_2, \\ & u \geqslant 0, \end{aligned} \tag{6}$$

where $u \in \mathbb{R}^m$ are the dual variables. Let (x_1^*, x_2^*) a feasible solution of (2). Then, x_2^* is the optimal solution to $\mathrm{LP}(x_1^*)$. Hence, it is a well-known result that there exists $u^* \in \mathbb{R}^m$ which solves $\mathrm{DLP}(x_1^*)$ and holds $d_2x_2 - u^t(A_1x_1 - b) = 0$. Thus, (x_1^*, x_2^*, u^*) is a feasible solution of (5). Similar arguments demonstrate the converse. □

Bearing in mind the previous theorem, in order to solve problem (2) we can solve the nonlinear problem (5). A usual technique to approach this kind of nonlinear problems consists in taking the duality gap, $d_2x_2 - u^t(A_1x_1 - b)$, as a penalty term and proceeding by analysing the associated problem. Nevertheless, the resulting function is not very tractable since it does not hold the properties we need to prove the exactness of the penalty method. Thus, we propose to take

$$p(x_1, x_2, u) = (d_2x_2 + b^t u - u^t A_1x_1)(\beta + c_{21}x_1 + c_{22}x_2) \geqslant 0 \quad \forall (x_1, x_2, u) \in S \times U$$

as the penalty function. Let us denote

$$F(x_1, x_2, u; \mu) = (\alpha + c_{11}x_1 + c_{12}x_2)(\beta + c_{21}x_1 + c_{22}x_2) + \mu p(x_1, x_2, u).$$

Hence, the penalty problem is given by

$$\begin{aligned} P(\mu): \quad \min_{x_1,x_2,u} \quad & F(x_1, x_2, u; \mu) \\ \text{s.t.} \quad & A_1x_1 + A_2x_2 \leqslant b, \\ & -u^t A_2 \leqslant d_2, \\ & x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad u \geqslant 0, \end{aligned} \tag{7}$$

where $\mu \geqslant 0$ is a scalar parameter. Notice that the feasible region of problem (7) includes sets of separate constraints in variables (x_1, x_2) and u. Let $\mu \geqslant 0$ fixed and $u \in U$, we define

$$\psi(u; \mu) = \min_{(x_1,x_2) \in S} \quad F(x_1, x_2, u; \mu). \tag{8}$$

Theorem 2. *For fixed μ, the function $\psi(u; \mu)$ is concave.*

Proof. This follows from the fact that the pointwise minimum of an arbitrary collection of concave functions defined on a convex set is concave [14]. □

Let us denote the sets of extreme points of S, U and $S \times U$ by $E(S)$, $E(U)$ and $E(S \times U)$, respectively.

Theorem 3. *For fixed μ, there exists $u^\star \in E(U)$ which solves the problem*

$$\min_{u \in U} \quad \psi(u; \mu). \tag{9}$$

Proof. The function $\psi(u;\mu)$ is bounded from below over U. Otherwise, for all $M \in \mathbb{R}$, a $u_M \in U$ exists such that $\psi(u_M;\mu) < M$. Hence, there is $(x_1^M, x_2^M) \in S$ so that

$$(\alpha + c_{11}x_1^M + c_{12}x_2^M)(\beta + c_{21}x_1^M + c_{22}x_2^M) + \mu p(x_1^M, x_2^M, u^M) < M.$$

Since $\mu p(x_1^M, x_2^M, u^M) \geqslant 0$, then

$$(\alpha + c_{11}x_1^M + c_{12}x_2^M)(\beta + c_{21}x_1^M + c_{22}x_2^M) < M,$$

which contradicts the fact that the function $(\alpha + c_{11}x_1 + c_{12}x_2)(\beta + c_{21}x_1 + c_{22}x_2)$ is bounded over the compact polyhedron S. As a consequence, taking also into account that U is a polyhedron, the problem (9) has a global minimum which is an extreme point of U [14]. □

Theorem 4. *For fixed* μ, *there exists* $(x_1, x_2, u) \in E(S) \times E(U)$ *which solves the problem* $P(\mu)$.

Proof.

$$\begin{aligned}\min_{(x_1,x_2,u)\in S\times U} F(x_1,x_2,u;\mu) &= \min_{u\in U}\ \min_{(x_1,x_2)\in S} F(x_1,x_2,u;\mu)\\ &= \min_{u\in E(U)}\ \min_{(x_1,x_2)\in S} F(x_1,x_2,u;\mu).\end{aligned}$$

Given $u \in E(U)$, $F(x_1, x_2, u; \mu)$ is a linear multiplicative function with positive factors, so it is quasiconcave [11]. Therefore, its minimum over the polyhedron S is reached at an extreme point. Hence

$$\begin{aligned}\min_{(x_1,x_2,u)\in S\times U} F(x_1,x_2,u;\mu) &= \min_{u\in E(U)}\ \min_{(x_1,x_2)\in E(S)} F(x_1,x_2,u;\mu)\\ &= \min_{(x_1,x_2,u)\in E(S)\times E(U)} F(x_1,x_2,u;\mu). \qquad \square\end{aligned}$$

Theorem 5. *A finite value* $\mu^\star$ *exists so that*, $\forall \mu \geqslant \mu^\star$, *any optimal solution of problem* $P(\mu)$ *is optimal to the LMLB problem* (2). *Moreover*, *an optimal solution of* $P(\mu)$, $\mu \geqslant \mu^\star$, *exists which is an extreme point of* $E(S) \times E(U)$ *and has a zero duality gap*.

Proof. Let $\{\mu_j\}$ be a nondecreasing sequence such that $\mu_j \to_{j\uparrow} \infty$. Let $(x_1^j, x_2^j, u^j) \in E(S) \times E(U)$ be an optimal solution to problem $P(\mu_j)$.

Since $E(S) \times E(U)$ is a finite set, there is an index j_0 and a point $(\overline{x}_1, \overline{x}_2, \overline{u}) \in E(S) \times E(U)$ so that $(x_1^j, x_2^j, u^j) = (\overline{x}_1, \overline{x}_2, \overline{u})$, $\forall j \geqslant j_0$.

Let $(\hat{x}_1, \hat{x}_2, \hat{u})$ be a feasible solution of problem (5). Then it is a feasible solution of problem (7), $\forall \mu$, and $d_2\hat{x}_2 - \hat{u}^t(A_1\hat{x}_1 - b) = 0$. Hence,

$$F(x_1^j, x_2^j, u^j; \mu) \leqslant F(\hat{x}_1, \hat{x}_2, \hat{u}; \mu) = (\alpha + c_{11}\hat{x}_1 + c_{12}\hat{x}_2)(\beta + c_{21}\hat{x}_1 + c_{22}\hat{x}_2).$$

In particular, for all $j \geqslant j_0$,

$$(\alpha + c_{11}\overline{x}_1 + c_{12}\overline{x}_2)(\beta + c_{21}\overline{x}_1 + c_{22}\overline{x}_2) + \mu_j p(\overline{x}_1, \overline{x}_2, \overline{u}) \leqslant (\alpha + c_{11}\hat{x}_1 + c_{12}\hat{x}_2)(\beta + c_{21}\hat{x}_1 + c_{22}\hat{x}_2).$$

Let $\mu_j \to \infty$. Since $p(\overline{x}_1, \overline{x}_2, \overline{u}) \geqslant 0$ and $(\hat{x}_1, \hat{x}_2, \hat{u})$ is fixed, then $d_2\overline{x}_2 - \overline{u}^t(A_1\overline{x}_1 - b) = 0$. Hence, $(\overline{x}_1, \overline{x}_2, \overline{u})$ is a feasible solution of problem (5) and

$$(\alpha + c_{11}\overline{x}_1 + c_{12}\overline{x}_2)(\beta + c_{21}\overline{x}_1 + c_{22}\overline{x}_2) \leqslant (\alpha + c_{11}\hat{x}_1 + c_{12}\hat{x}_2)(\beta + c_{21}\hat{x}_1 + c_{22}\hat{x}_2).$$

As a consequence, $(\overline{x}_1, \overline{x}_2, \overline{u})$ is an optimal solution of problem (5).

Let $\mu^\star = \mu_{j_0}$. Let $(x_1^\mu, x_2^\mu, u^\mu)$ be an optimal solution of $P(\mu)$, $\mu > \mu^\star$. Since the penalty function is a nonincreasing function of μ [3], $p(x_1^\mu, x_2^\mu, u^\mu) \leqslant p(\overline{x}_1, \overline{x}_2, \overline{u}) = 0$. Therefore, $d_2x_2^\mu - (u^\mu)^t(A_1x_1^\mu - b) = 0$ and $(x_1^\mu, x_2^\mu, u^\mu)$ is an optimal solution of (5). By applying Theorem 1, it is an optimal solution of the LMLB problem.

On the other hand, since $(\overline{x}_1, \overline{x}_2, \overline{u})$ and $(x_1^\mu, x_2^\mu, u^\mu)$, $\forall \mu > \mu^\star$ are optimal solutions of (5), then $F(\overline{x}_1, \overline{x}_2, \overline{u}; \mu) = F(x_1^\mu, x_2^\mu, u^\mu; \mu)$. Hence, $(\overline{x}_1, \overline{x}_2, \overline{u}) \in E(S) \times E(U)$ is an optimal solution of $P(\mu)$. □

Example 1. To get an insight of the meaning of previous theorems we consider the problem:

$$\begin{aligned} \min_{x_1} \quad & (25 - x_1)(1 + x_2) \quad \text{where } x_2 \text{ solves} \\ \min_{x_2} \quad & -x_2 \\ \text{s.t.} \quad & (x_1, x_2) \in S, \end{aligned} \tag{10}$$

where $S=\{(x_1, x_2) : x_1 - x_2 \leqslant 10;\ 3x_1 + 2x_2 \leqslant 55;\ x_1 + 4x_2 \geqslant 20;\ x_1 - x_2 \geqslant -15;\ 6x_1 - x_2 \geqslant -5;\ x_1, x_2 \geqslant 0\} \subset \mathbb{R}^2$. For each $x_1 \in S_1 = [0, 15]$, a direct computation shows that the optimal solution of $\mathrm{LP}(x_1)$ is

$$x_2 = M(x_1) = \begin{cases} 6x_1 + 5 & \text{if } 0 \leqslant x_1 \leqslant 2, \\ x_1 + 15 & \text{if } 2 \leqslant x_1 \leqslant 5, \\ \dfrac{55 - 3x_1}{2} & \text{if } 5 \leqslant x_1 \leqslant 15. \end{cases}$$

For each $x_1 \in S_1$, the dual feasible region is $U = \{(u_1, u_2, u_3, u_4, u_5) \in \mathbb{R}^5 : u_1 - 2u_2 + 4u_3 - u_4 - u_5 \leqslant -1;\ u_i \geqslant 0, i = 1, \ldots, 5\}$. The penalty problem $P(\mu)$ is

$$\begin{aligned} P(\mu) : \min \quad & (25 - x_1)(1 + x_2) + \mu(-x_2 - (x_1 - 10)u_1 - (3x_1 - 55)u_2 \\ & \quad - (20 - x_1)u_3 + (15 + x_1)u_4 + (5 + 6x_1)u_5)(1 + x_2) \\ \text{s.t.} \quad & (x_1, x_2) \in S, \quad (u_1, u_2, u_3, u_4, u_5) \in U. \end{aligned}$$

According to Theorem 4, we enumerate $E(S) \times E(U)$ in order to get the optimal solution of $P(\mu)$

$$(x_1^*, x_2^*, u_1^*, u_2^*, u_3^*) = \begin{cases} (12, 2, 0, 0.5, 0, 0, 0) & \text{if } \mu \in [0, 14/15], \\ (15, 5, 0, 0.5, 0, 0, 0) & \text{if } \mu \geqslant 14/15. \end{cases}$$

Therefore, $\mu^\star = \frac{14}{15}$ and the optimal solution of (10) is (15, 5).

Remark 6. An important consequence of Theorem 5 is that there is an extreme point of S which is an optimal solution of the LMLB problem. Hence, an examination of all extreme points of S provides an algorithm that will find the optimal solution in a finite number of steps. However, except for very simple examples, this technique is not very promising because of the generally large number of extreme points of a polyhedron.

Like linear bilevel or linear fractional bilevel problems [5,7], LMLB problems are quasiconcave bilevel problems [6]. But, unlike them, the Kth-best algorithm, a more successful enumeration scheme, does not work, as the following example shows:

$$\begin{aligned} \min \quad & (50 - 2x_1 - 3x_2)(2 + x_1 + x_2) \quad \text{where } x_2 \text{ solves} \\ \min \quad & -x_2 \\ \text{s.t.} \quad & (x_1, x_2) \in S, \end{aligned} \tag{11}$$

where $S = \{(x_1, x_2) : -5x_1 - 3x_2 \leqslant -15;\ -x_1 + 4x_2 \leqslant 28;\ 2x_1 + 3x_2 \leqslant 32;\ x_1 + x_2 \leqslant 13;\ 2x_1 - x_2 \leqslant 13;\ x_1 - 4x_2 \leqslant 3;\ x_1, x_2 \geqslant 0\}$. By simply checking the extreme points, we get the optimal solution (4, 8), $f_1(4, 8) = 252$.

The Kth-best algorithm essentially asserts that, starting with an optimal solution of problem (4), an optimal solution of (2) can be obtained by getting the best of the extreme points adjacent to all previously analysed extreme points which is a point of IR. The optimal solution of the corresponding problem (4) is (3, 0), $f_1(3, 0) = 220$. Its adjacent extreme points are $(0, 5) \notin \mathrm{IR}$, $f_1(0, 5) = 245$ and $(7, 1) \notin \mathrm{IR}$, $f_1(7, 1) = 330$. Taking now (0, 5), its adjacent extreme points are $(0, 7) \in \mathrm{IR}$, $f_1(0, 7) = 261$ and (3, 0), which has just been considered. Therefore, according with the Kth-best algorithm, (0, 7) would be the optimal solution, which is not true.

Another difference with linear bilevel and linear fractional bilevel problems [7,12] is that the optimal solution of LMLB problems is not necessarily a boundary feasible extreme point. According to [12], a point $x \in \mathrm{IR}$ is a boundary feasible extreme point if there exists an edge E of S such that x is an extreme point of E, and the other extreme point of E is not an element of IR. Notice that the optimal solution of this example (4, 8) is not a boundary feasible extreme point as it has both edges in IR.

Step 1.
Set $l = 0$.
Let (x_1^0, x_2^0) be an optimal solution of (4).
Let x_2^* be the optimal solution of LP (x_1^0).
If $x_2^0 = x_2^*$, stop; (x_1^0, x_2^0) is a global optimum of (2).
Otherwise, set $(\tilde{x}_1, \tilde{x}_2) = (x_1^0, x_2^*) \in IR$.
Let y be an optimal solution of (12). Go to Step 2.
Step 2.
Select $\hat{y}$ the best adjacent extreme point to y in IR w.r.t. f_1.
Step 3.
If $f_1(\hat{y}) < f_1(y)$, set $y = \hat{y}$. Go to Step 2.
Step 4.
Set $l = l+1$, $x^l = y$, $\gamma_l = f_1(x^l)$.
Step 5.
Set $h = 1$, $w^h = x^l$.
Step 6.
Find $\pi(x - w^h) \geq 1$, a γ_l-valid cut for (f_1, IR).
Step 7.
Let υ be an optimal solution of (13).
If $\pi(\upsilon - w^h) \leq 1$, stop; x^l is a global solution of the LMLB problem.
Otherwise, set $S = S \cap \{x \in \mathrm{R}^n : \pi(x - w^h) \geq 1\}$ and
$IR = IR \cap \{x \in \mathrm{R}^n : \pi(x - w^h) \geq 1\}$.
Step 8.
Select $\hat{\upsilon}$ the best adjacent extreme point to υ in IR w.r.t. f_1.
Step 9.
If $f_1(\hat{\upsilon}) < f_1(\upsilon)$, set $\upsilon = \hat{\upsilon}$ and go to Step 8.
Step 10.
If $f_1(\upsilon) \geq \gamma_l$, set $h = h+1$, $w^h = \upsilon$ and go to Step 6.
Otherwise, set $y = \upsilon$ and go to Step 4.

Fig. 1. LSCP algorithm.

On the other hand, since linear multiplicative functions with positive factors are quasiconcave, problem (2) is a particular case of the bilevel quasiconcave problem [6]. Therefore, IR consists of the union of connected faces of the polyhedron S.

The aim of the algorithm developed in the next section is to efficiently search between extreme points of S, using the underlying ideas in the Horst and Tuy [9] pure cutting algorithm for solving concave minimisation problems and taking into account the geometry of IR. From now on, in order to avoid cumbersome notation, when possible, we will write x instead of (x_1, x_2).

3. The LSCP algorithm

After getting an initial feasible extreme point, the typical iteration of the algorithm looks for a local vertex-minimum point, i.e., an extreme point in IR with a better value of the upper level objective function than any of its adjacent extreme points. Next, in order to check if it is a global optimum, cutting planes are successively introduced until the global optimality of the incumbent extreme point is proved or a better extreme point in IR is found. The outline of the algorithm is given in Fig. 1. Next, we briefly describe the main phases of the algorithm.

3.1. Initialisation

The initialisation phase includes Steps 1–3. Firstly, it checks if the problem considered is trivial, i.e., the optimal solution to problem (4) is a point of IR. If so, the algorithm terminates. Otherwise, we have $\tilde{x} = (\tilde{x}_1, \tilde{x}_2) \in$ IR. Since we are specifically interested in extreme points, by solving the linear multiplicative problem (12), we get the best extreme point y on the face of the polyhedron S that contains $\tilde{x}$,

$$\begin{aligned} \min_x \quad & (\alpha + c_{11}x_1 + c_{12}x_2)(\beta + c_{21}x_1 + c_{22}x_2) \\ \text{s.t.} \quad & H_i(x) = 0, \ i \in I, \\ & H_i(x) \leqslant 0 \quad \text{otherwise,} \end{aligned} \tag{12}$$

where $H_i(x) \leqslant 0, i = 1, \ldots, m + n_1 + n_2$, stands for the ith constraint defining S and I is the set of indices of constraints that are binding at $\tilde{x}$. Since the objective function is quasiconcave, there is an optimal solution of (12) which is an extreme point. Moreover, since the whole face is in IR, this is an extreme point of IR. From this extreme point, the local search starts. In Step 2, its adjacent extreme points in IR are investigated to look for the one with the best value of the upper level objective function f_1. If this value is better than $f_1(y)$, in Step 3 we select this extreme point and repeat the process. Otherwise, the incumbent extreme point y is declared a local vertex-minimum extreme point, in short, a local optimal solution. The purpose of the following steps is to find an extreme point of IR with a better value of f_1 than the incumbent extreme point. Otherwise, we conclude that it is a global optimum.

In Step 1, we can also take any of the extreme points on that face by solving (12) replacing the objective function by a linear one. This would avoid having to solve a multiplicative problem.

3.2. Cutting plane construction

Let x^l be the incumbent extreme point and $\gamma_l = f_1(x^l)$. We may restrict our search for a better point to the subset of IR consisting of points x holding $f_1(x) < \gamma_l$. Two main problems arise when doing that. Firstly, IR is implicitly defined and no explicit expression is known. We only know that IR consists of the union of connected faces of the polyhedron S. Secondly, if we restrict the search by adding the constraint $(\alpha + c_{11}x_1 + c_{12}x_2)(\beta + c_{21}x_1 + c_{22}x_2) < \gamma_l$ to the feasible set, the resulting region will no longer be a polyhedron and so, in general, it will be difficult to be handled. In order to preserve the polyhedral structure of the constraint set, we consider the approach by Horst and Tuy [9] and construct an affine function $l(x)$ such that the constraint $l(x) \geqslant 0$ does not exclude any point $x \in$ IR with $f_1(x) < \gamma_l$. A linear inequality $l(x) \geqslant 0$ with this property is called a γ_l-valid cut for (f_1, IR) [9].

Assume for the time being that x^l is nondegenerate. Note that we cannot directly construct the cutting plane as indicated in [9] since x^l is a local optimum only with respect to extreme points in IR. Hence, there could be adjacent extreme points to x^l (obviously, not in IR) with a better value of f_1.

Let $z^i \in$ IR, $i = 1, \ldots, n$, be the adjacent extreme points to x^l. We assume that $z^i \in$ IR, $i = 1, \ldots, r$ and $z^i \notin$ IR, $i = r+1, \ldots, n$. We can only assure that $f_1(x^l) \leqslant f_1(z^i), i = 1, \ldots, r$. Since $z^1 - x^l, \ldots, z^n - x^l$ are linearly independent, there is a unique hyperplane $\pi(x - x^l) = 1$ passing through the n points $z^1, \ldots, z^n$, where $\pi = eQ^{-1}$, e is a n-row vector of ones and $Q : n \times n$ is the matrix whose ith column is $z^i - x^l$.

Theorem 7. *The half-space* $eQ^{-1}(x - x^l) \geqslant 1$ *defines a* γ_l*-valid cut for* (f_1, IR).

Proof. Let $P = \text{conv}(x^l, z^1, \ldots, z^n)$, where conv refers to the convex hull. Let K be the cone vertexed at x^l and generated by the half-lines emanating from x^l in the directions $z^i - x^l, i = 1, \ldots, n$. Clearly $P = K \cap \{x : eQ^{-1}(x - x^l) \leqslant 1\} \subset S$.

On the other hand, as f_1 is quasiconcave, $\{x : f_1(x) \geqslant \gamma_l\}$ is convex. Moreover, $f_1(x^l) = \gamma_l$ and $f_1(z^i) \geqslant \gamma_l, i = 1, \ldots, r$, thus $P \cap \text{IR} \subset \{x : f_1(x) \geqslant \gamma_l\}$. Hence, if $x \in \text{IR} \subset K$ satisfies $f_1(x) < \gamma_l$, we must have $x \notin P$, that is to say $eQ^{-1}(x - x^l) > 1$.

Therefore, $\{x \in \text{IR} : f_1(x) < \gamma_l\} \subset \{x \in \text{IR} : eQ^{-1}(x - x^l) > 1\}$ proving that $eQ^{-1}(x - x^l) \geqslant 1$ defines a γ_l-valid cut for (f_1, IR). □

When x^l is degenerate, it may have more than n adjacent extreme points, $z^1, \ldots, z^s$. The method proposed by Carvajal–Moreno described in Horst and Tuy [9, pp. 97–99] can be used. A normal vector π is determined as a basic solution of the system of inequalities $\pi d^i \geqslant 1/\alpha_i, \ i = 1, \ldots, s$, where $d^1, d^2, \ldots, d^s, \ (s > n)$ are directions of the edges of S emanating from x^l, and $\alpha_i = 1$. Then $\pi(z^i - x^l) \geqslant 1$ is a γ_l-valid cut.

3.3. Termination test

Theorem 8. *Let x^l be an extreme point, which is a local optimal solution of problem* (2) *with $f_1(x^l)=\gamma_l$. Let $\pi(x-x^l)\geqslant 1$ be the γ_l-valid cut. If the optimal value of the problem*

$$\max_{x\in \mathrm{IR}} \pi(x-x^l) \tag{13}$$

is less or equal to 1, *then x^l is a global optimal solution of the LMLB problem.*

Proof. Since $\{x \in \mathrm{IR} : f_1(x) < \gamma_l\} \subset \{x \in \mathrm{IR} : \pi(x-x^l) > 1\}$, if $\pi(x-x^l) \leqslant 1$, $\forall x \in \mathrm{IR}$ then $f_1(x) \geqslant \gamma_l$, $\forall x \in \mathrm{IR}$. Therefore x^l is a global optimal solution of problem (2). □

Hence, by applying this theorem, Step 7 allows us to check if x^l is a global optimum. Otherwise, a search for another local optimum starts in the polyhedron resulting from 'cutting' the region of no interest of the current polyhedron.

3.4. Algorithm convergence

Theorem 9. *The algorithm terminates at a global optimum of the LMLB problem.*

Proof. Each time an iteration of the algorithm is completed, i.e., the algorithm returns to Step 4, a better extreme point in IR is obtained. Since the number of extreme points in IR is finite, eventually an extreme point which is a global optimum will be obtained.

Moreover, in every iteration, each time a cut is applied, the region of no interest which is eliminated contains an extreme point of IR. Again, since the number of extreme points in IR is finite, eventually either the algorithm stops or finds a better extreme point in IR. □

This theorem shows that the proposed method generates a convergence sequence to a global optimal solution. Now, a few remarks about the complexity of the algorithm should be made. In Step 1 we must solve the linear multiplicative problem (4). For this purpose, a parametric simplex algorithm developed by Konno and Kuno [10] can solve linear multiplicative problems in no more than twice as much computation time than that of solving a linear program.

The LSCP algorithm, when solving a nontrivial LMLB problem, basically consists of two main phases. The first one involves computing adjacent extreme points to the incumbent one, which can be done, for instance, by applying pivoting operations. Matheiss and Rubin [13] give a survey and comparison of methods for finding all vertices of a polyhedral set. The second main phase involves solving the linear bilevel problem (13). Linear bilevel problems have been proved to be strongly NP-hard. A good insight into methods globally solving them is given in [2]. For medium sized problems, the Kth-best algorithm mentioned in Remark 6 provides acceptable results. Furthermore, it is also worth pointing out that, in all steps of the LSCP algorithm, only linear programming problems have to be solved. The following example confirms the applicability and the convergence of the algorithm.

3.5. An application of the algorithm: Example 2

For illustrating the algorithm, we consider the problem (11), replacing the upper level objective function by $f_1 = (90-5x_1-3x_2)(2+x_1+x_2)$. The polyhedron S and IR are shown in Fig. 2. The algorithm proceeds as follows:

Step 1: The optimal solution of problem (4) is $(3, 0) \notin \mathrm{IR}$. The optimal solution of problem LP(3) is $\tilde{x}=(3, \frac{31}{4}) \in \mathrm{IR}$, which is not an extreme point of the polyhedron S. The optimal solution of (12) is $y=(0, 7)$.

Step 2: The best adjacent extreme point to y in IR is $(4, 8)$.

Step 3: $f_1(4, 8) = 644 \not< 621 = f_1(0, 7)$.

Step 4: $l=1$, $x^1=(0, 7)$, $\gamma_1 = 621$.

Step 5: $h=1$, $w^1=(0, 7)$. *Step* 6: The cut is $\frac{3}{8}x_1 - \frac{4}{8}(x_2-7) \geqslant 1$.

Step 7: $v=(\frac{26}{3}, \frac{13}{3})$, $\max = \frac{55}{12} \not\leqslant 1$.

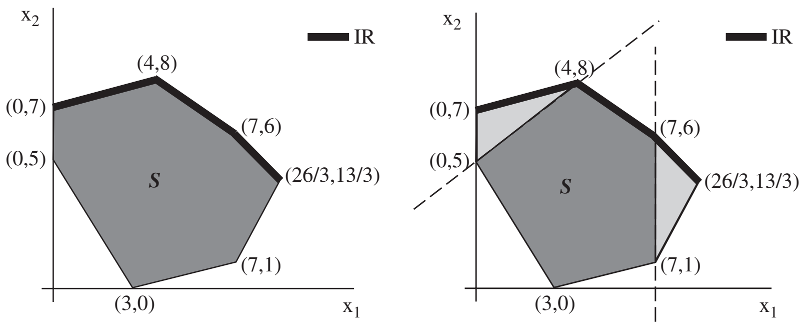

Fig. 2. Common constraint region S and feasible region IR of Example 2.

$$S = \mathrm{conv}((0,5),(4,8),(7,6),(26/3,13/3),(7,1),(3,0)),$$
$$\mathrm{IR} = \mathrm{conv}((4,8),(7,6)) \cup \mathrm{conv}((7,6),(26/3,13/3)).$$

Step 8: The best adjacent extreme point in IR is (7, 6).
Step 9: $f_1(7,6) = 555 \not< 505 = f_1(\frac{26}{3}, \frac{13}{3})$.
Step 10: Since $f_1(\frac{26}{3}, \frac{13}{3}) = 505 < \gamma_1 = 621$, we make $y = (\frac{26}{3}, \frac{13}{3})$.
Step 4: $l = 2$, $x^2 = (\frac{26}{3}, \frac{13}{3})$, $\gamma_2 = 505$.
Step 5: $h = 1$, $w^1 = (\frac{26}{3}, \frac{13}{3})$.
Step 6: The cut is $-\frac{3}{5}(x_1 - \frac{26}{3}) \geqslant 1$.
Step 7: $v = (4,8)$, max $= \frac{14}{5} \not\leqslant 1$.

$$S = \mathrm{conv}((0,5),(4,8),(7,6),(7,1),(3,0)),$$
$$\mathrm{IR} = \mathrm{conv}((4,8),(7,6)).$$

Step 8: The best adjacent extreme point to v in IR is (7, 6).
Step 9: $f_1(7,6) = 555 < 644 = f_1(4,8)$, so we make $v = (7,6)$.
Step 8: The best adjacent extreme point to v in IR is (4, 8).
Step 9: $f_1(4,8) = 644 \not< 555 = f_1(7,6)$.
Step 10: Since $f_1(7,6) = 555 \geqslant \gamma_2 = 505$, we set $h = 2$, $w^2 = (7,6)$.
Step 6: The cut is $-\frac{7}{15}(x_1 - 7) - \frac{3}{15}(x_2 - 6) \geqslant 1$.
Step 7: $v = (4,8)$, max $= 1 \leqslant 1$, hence, $x^2 = (\frac{26}{3}, \frac{13}{3})$ is a global optimum.

4. Extending optimality results

Next, we consider LMLB problems with upper level constraints involving both level variables (GLMLB problems). They can be formulated as

$$\begin{aligned} \min_{x_1} \quad & (\alpha + c_{11}x_1 + c_{12}x_2)(\beta + c_{21}x_1 + c_{22}x_2) \\ \text{s.t.} \quad & B_1x_1 + B_2x_2 \leqslant b_1, \end{aligned}$$

where x_2 solves

$$\begin{aligned} \min_{x_2} \quad & d_2x_2 \\ \text{s.t.} \quad & A_1x_1 + A_2x_2 \leqslant b, \\ & x_1 \geqslant 0, \quad x_2 \geqslant 0, \end{aligned} \tag{14}$$

where $B_1 : m_1 \times n_1$; $B_2 : m_1 \times n_2$ and $b_1 \in \mathbb{R}^{m_1}$. Let T be the polyhedron defined by the whole set of constraints. We assume that T is a nonempty bounded polyhedron and IR is nonempty to guarantee the existence of a solution to the problem. If S is not bounded, we assume that the lower level problem (3) achieves its minimum, for all $x_1 \in S_1$.

Bilevel problems are very sensitive to the addition of these kind of constraints. For instance, if we put the constraint $-x_1+12x_2 \leqslant 20$ in the upper level of Example 1, then IR $=\emptyset$. On the other hand, if we add the constraint $-x_1+2x_2 \leqslant 19$ then IR $=$ conv((0, 5), (1, 10)) $\cup$ conv((9, 14), (15, 5)), i.e., the inducible region would no longer be connected, but the optimal solution would continue to be an extreme point of the whole set of constraints. The following theorems, whose proofs are similar to the corresponding results in Section 2 by properly changing sets S and T, establish the latter property.

Problem (14) can be reformulated as the equivalent one-level problem:

$$\begin{aligned} \min_{x_1,x_2,u} \quad & (\alpha + c_{11}x_1 + c_{12}x_2)(\beta + c_{21}x_1 + c_{22}x_2) \\ \text{s.t.} \quad & B_1x_1 + B_2x_2 \leqslant b_1, \\ & A_1x_1 + A_2x_2 \leqslant b, \\ & -u^tA_2 \leqslant d_2, \\ & d_2x_2 - u^t(A_1x_1 - b) = 0, \\ & x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad u \geqslant 0. \end{aligned} \tag{15}$$

The penalty problem we are interested in is given by

$$\begin{aligned} P_G(\mu): \quad \min_{x_1,x_2,u} \quad & F(x_1, x_2, u; \mu) \\ \text{s.t.} \quad & B_1x_1 + B_2x_2 \leqslant b_1, \\ & A_1x_1 + A_2x_2 \leqslant b, \\ & -u^tA_2 \leqslant d_2, \\ & x_1 \geqslant 0, x_2 \geqslant 0, u \geqslant 0. \end{aligned} \tag{16}$$

Let $\mu \geqslant 0$ fixed and $u \in U$, we define

$$\psi_G(u; \mu) = \min_{(x_1,x_2)\in T} F(x_1, x_2, u; \mu). \tag{17}$$

Theorem 10. *For fixed* μ, *the function* $\psi_G(u; \mu)$ *is concave.*

Theorem 11. *For fixed* μ, *there exists* $u^\star \in E(U)$ *which solves the problem* $\min_{u\in U} \psi_G(u; \mu)$.

Theorem 12. *For fixed* μ, *there exists* $(x_1, x_2, u) \in E(T) \times E(U)$ *which solves the problem* $P_G(\mu)$.

Theorem 13. *A finite value* $\mu^\star$ *exists so that*, $\forall \mu \geqslant \mu^\star$, *any optimal solution of problem* $P_G(\mu)$ *is optimal to the GLMLB problem* (2). *Moreover, there exists an optimal solution of* $P_G(\mu)$, $\mu \geqslant \mu^\star$, *which is an extreme point of* $E(T) \times E(U)$ *and has a zero duality gap.*

5. Conclusions and future work

In this paper we have analysed the linear multiplicative/linear bilevel problem when there are upper level constraints involving both level variables and when not. We have proved that, in both cases, the optimal solution is achieved at an extreme point of the region defined by the whole set of constraints. Also we have shown, with examples, several specificities which make LMLB problems different from linear bilevel and linear fractional bilevel ones and more difficult to be solved. Finally, an algorithm has been proposed that combines the search amongst extreme points of S with the construction of valid cuts to efficiently obtain the global optimum.

On the other hand, when solving linear multiplicative one level problems over polyhedra, a very promising approach is to use an outcome-space reformulation of the problem. The LMLB problem (2) can be written as

$$\begin{aligned} \min \quad & (\alpha + c_{11}x_1 + c_{12}x_2)(\beta + c_{21}x_1 + c_{22}x_2) \\ \text{s.t.} \quad & (x_1, x_2) \in \operatorname{conv}(E(\mathrm{IR})). \end{aligned}$$

Let $Z_1 = \alpha + c_{11}x_1 + c_{12}x_2$ and $Z_2 = \beta + c_{21}x_1 + c_{22}x_2$. Through Z_1, Z_2, every point $(x_1, x_2) \in \operatorname{conv}(E(\mathrm{IR}))$ maps into a point $(z_1, z_2) \in \mathbb{R}^2$. Let Z_{IR} be the image of $\operatorname{conv}(E(\mathrm{IR}))$ by this map. We consider the following bicriteria linear problem:

$$\begin{aligned} \min_{x_1, x_2} \quad & Z = (Z_1, Z_2) \\ \text{s.t.} \quad & (x_1, x_2) \in \operatorname{conv}(E(\mathrm{IR})). \end{aligned} \tag{18}$$

So, we can readily derive from Theorem 1 in Aneja et al. [1], that a global optimal solution of problem (2) is attained at (x_1^*, x_2^*), which maps into an efficient extreme point of Z_{IR}. In other words, efficient solutions are those solutions for which none of the criteria can be improved without deterioration of the other criterion. It remains for future work to explore this approach.

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